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A TALE OF TWO TREATISES (SAGA OF THE CONQUEST OF MT EVEREST OF NUMBER THEORY)

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Abstract: Diophantus Arithmetica-Fermat's Marginal note- Marvellous proof - No space to write the details of proof-Futile attempts by eminent mathematicians for over 350 years- E.T. Bell's Book- The Last Problem motivates Andrew Wiles- Working for seven years on the attic of his house - Finally FLT is proved in 1995 with some participation of Richard Taylor.

Keywords and Phrases: Diophantus - Arithmetica - Pythagoras Theorem - Fermat's Last Theorem (FLT) -E.T. Bell's Book - The Last Problem -Andrew Wiles -Elliptic Curves - Modular Forms - Taniyama -Shimura Conjecture - Richard Taylor.

1. Introduction

I

William Durham said, "Students of Literature read Shakespeare, students of Music listen to Bach. But this tradition of studying the major works of the masters is, if not wholly absent, certainly uncommon in Mathematics".

Laplace exhorted us all, "Read Euler, Read Euler, He is the master of us all"

When asked how he developed his mathematical abilities, Abel replied, "By studying the masters, not their pupils"

Fermat's Last Theorem-FLT, was first conjectured by Pierre de Fermat in the year 1637 in the form of a passing comment in the margin of a page in his personal copy of the Book *Arithmetica* of Greco-Alexandrian Mathematician of the third century C.E. The first proof was released in 1993 by the Princeton University Professor Andrew Wiles and published in 1995 after its fine tuning. It took about 358 years of effort by mighty mathematicians of the world to resolve the riddle.

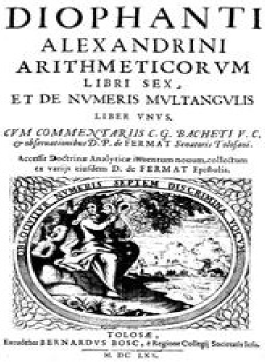
Heinrich Olbers, a German Astronomer and admirer of C F Gauss, once wrote to Gauss, encouraging him to compete for a prize that had been offered by the Paris Academy for a solution of Fermat's challenge. "It seems to me dear Gauss, that you should get busy about this" . Gauss wrote back, "I am very much obliged for your news concerning the Paris Prize, but I confess that Fermat's Last Theorem as an isolated proposition, has very little interest for me, for I could easily lay down a multitude of such propositions which one could neither prove nor disprove."

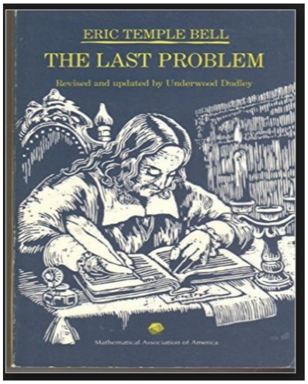
David Hilbert was asked why he never attempted a proof of FLT. He replied, "Before beginning I should have to put in three years of intensive study, and I have not that much of time to squander on a probable failure".

Euclid had interest in Number Theory, but it was not his seminal contribution to Mathematics. His main interest was geometry and his master piece was Elements presented in 13 Books. Books 7, 8, 9 and 10 deal with elementary number theory.

Diophantus of Alexandria (201- 285 CE) wrote the book *Arithmetica*. This was also a compendium of 13 Books, only 6 of them were extant. They were published. The Latin translation of Claude Gasper Bachet published in 1621 came to the hands of Fermat. Diophantus, hailed as Father of Algebra, was of Greek origin. His book *Arithmetica*, is a unique collection of 189 problems of algebra and their solutions. According to some available written sources it is opined that he must have lived after 150 BCE and before 350 CE. With some more evidence, we are led to believe that Diophantus lived in Alexandria around 250 CE.

Many scholars have studied *Arithmetica*. The work deals with linear and non-linear equations, determinate and indeterminate equations, which are the pinnacles of Greek Algebra. Diophantus was the first Greek mathematician who recognised fractions as numbers, thus allowed positive rational numbers for the coefficients and solutions. We all know that one of the greatest mathematical challenges of all times -The Fermat's Last Theorem, arose from the Book of Diophantus. It had been for three and half centuries, a notorious, epoch making, tantalizing, historic and emotional puzzle. E.T. Bell said, that he, (Diophantus), accomplished what he did with available techniques, places him beyond question, among the great algebraists.





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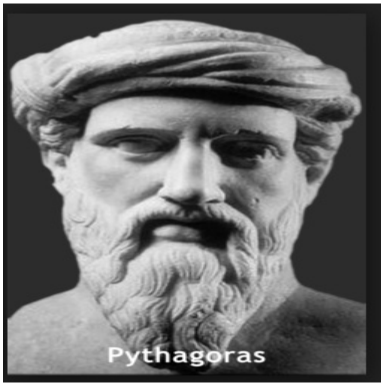

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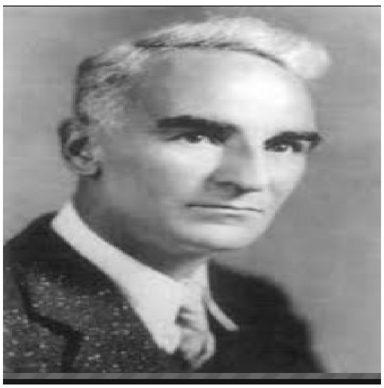

E.T. Bell (1883-1960) Andrew Wiles (1953)

Not much is known of the life and other facts about Diophantus. Just one detail carved on his tomb stone in the form of a riddle is available.

"God granted him to be a boy for the sixth part of his life, and adding a twelfth part to this, He clothed his cheeks with down; He lit him the light of wedlock after a seventh part, and five years after his marriage He granted him a son. Alas! late-born wretched child; after attaining the measure of half his father's full life, chill Fate took him. After consoling his grief by this science of numbers for four years, he ended his life".

Solution to this riddle:

$$\begin{gathered}\text{Suppose the span of life of Diophantus is x years}\\ \text{Period of boyhood} \rightarrow x/6\\ \text{Period of youth} \rightarrow x/12\\ \text{Period elapsed before wedlock} \rightarrow x/7\\ \text{A son was born after 5 years}\\ \text{Son's life span} \rightarrow x/2\\ \text{There after he lived for 4 years}\\ \text{Therefore } x = x/6 + x/12 + x/7 + 5 + x/2 + 4\\ \text{Solving this equation we get } x = 84 \text{ years}\end{gathered}$$

Most of the Books in the valuable collection of the Great Alexandrian Library was destroyed. It is indeed a miracle that some books somehow managed to survive fully or partially. Just 6 out of 13 volumes of *Arithmetica* survived. In these volumes we can find the entire knowledge of numbers as one finds the treasure of knowledge of geometry earlier to Euclid in Elements.

Hypatia (370-415 AD), was a distinguished mathematician of Alexandria, known for her works on Euclid's Elements, Diophantus's *Arithmetica* and such other works of ancient Greece. She worked with her father Theon, who was also a reputed mathematician of Greek origin, flourished in Alexadria. She wrote good commentaries on Euclid, Diophantus, Apollonius, Ptolemy and such others. What the mathematical world knows about Diophantus is primarily from the works of Theon and Hypatia. According to P. Tannery, that all existing manuscripts known to him were derived from a common source and that that source was Hyaptia's commentary. The mathematical world today owes Hypatia a great debt, for without her, we would not have got the treasure of Diophantus *Arithmetica.*

The *Arithmetica* contains scores of problems and for each one, Diophantus gives detailed solution. Fermat was inspired by this Book. While reading through the various problems and solutions, Fermat would scribble down his considered comments. While reading Book II of *Arithmetica*, Fermat encountered series of problems and their solutions concerning Pythagoras Theorem and Pythagorean triples. He recalled earlier writings of Euclid on Pythagorean triples and their availability in abundance. "Suddenly, in a moment of genius that would immortalize the Prince of Amateurs, he created an equation that, though very similar to Pythagoras's equation had no solutions at all". This was the equation that ignited the mind of ten year old Andrew Wiles who found it in the Book: *The Last Problem* by E.T. Bell. Wiles would recall, thirty years later, "It looked so simple, and yet all the great mathematicians in history couldn't solve it. Here was a problem that I, a ten year old, could understand and I knew from that moment that I would never let go. I had to solve it".

The Last Problem by E.T. Bell is a popular account of the origins of FLT. This essay is the tale of two Treatises: 1. *Arithmetica* and 2. *The Last Problem*. The first Book inspired Fermat and the second inspired Wiles. An epoch making tale followed. The tale commenced in the year 1637 in Toulouse in France and concluded in 1993 in the premises of the University of Cambridge, England. It was formally published in 1995, after filling up some gaps and fine tuning.

II

While reading through the pages of *Arithmetica* concerning Pythagorean Triples, Fermat suddenly thought of something that evaded the scholars of ancient Greece. From the equation

$$x^2 + y^2 = z^2$$

Fermat jumped to the equation

$$x^3 + y^3 = z^3$$

On further examining Fermat found that this new equation $x^3 + y^3 = z^3$ had no integer solution. He was greatly surprised- "Could it really be the case : that this minor modification turned Pythagoras equation rich with infinite number of integer solutions, into an equation with no solution?". He went one step further and considered the equation $x^4 + y^4 = z^4$ whose fate was also the same. He then conjectured that there appeared to be no three integers x, y and z such that $x^n + y^n = z^n$, n=3, 4, 5 In the margin of his personal copy of *Arithmetica*, next to the problem II (8), Fermat recorded his comment:

It is impossible for a cube to be written as a sum of two cubes or a fourth power to be written as the sum of two fourth powers or, in general, for any number which is a power greater than the second to be written as sum of two like powers.

Thus he asserted that there cannot be a Fermatean triple, and the riddle arose when he wrote his next note: *"I have a truly marvelous demonstration of this proposition, which this margin is too narrow to contain"*. The marginal note which became the notorious challenge for centuries in the world of mathematics, as Fermat's Last Theorem, was one of the many inspirational thoughts scribbled in the book.

Fermat's personal copy of the Book of Diophantus that is *Arithmetica*, has not survived. He seems to have written this marginal note around 1637 A.D. This note surfaced in print in the year 1670 at the instance of Fermat's eldest son Clement Samuel.

Fermat died in January- 1665, and Fermat's writings were at the risk of being lost forever. Fortunately Samuel appreciated the need to preserve the writings of his father. The mathematical world should be thankful to him. "Samuel spent five years collecting his father's notes and letters, and examining the jottings in the margins of his personal copy of *Arithmetica*".

"Samuel undertook to publish these annotations in a special edition of Arithmetica. In 1670, he brought out Diophantus's Arithmetica containing observations by P. de Fermat. Alongside Bachet's original Greek and Latin translations were forty eight observations made by Fermat.... If it were not for Clement Samuel, the enigma known as Fermat's Last Theorem would have died with its creator"

With no further information about the details of the so called demonstration, Fermat's conjecture attained the notoriety and became famous all over the world known as Fermat's last Theorem, an unsolved problem haunting the best brains of mathematicians over centuries!

Mighty minds tried and failed. In the course of meritorious and praise worthy unsuccessful attempts, newer branches of Number Theory and related areas sprouted. FLT deserves a unique place in the history of mathematics. Because of its simplicity, it has attracted and tantalized amateurs and professionals alike. "Its remarkable fecundity" has led to the development of large areas of mathematics such as, algebraic number theory , ring theory , ideals of Kummer, algebraic geometry in the later part of 19th century and, theory few in the 20th century. It is as if some super mind planned it, all over the centuries have been developing diverse streams of thought only to have them fuse in a spectacular synthesis to resolve FLT. No singular brain can claim expertise in all the ideas that have gone into this marvelous proof. "In this age of specialization each one of us knows more and more about less and less".

Several Institutions announced prizes for the successful resolution of the tantalizing riddle. In 1816 and again in 1850, the French Academy of Sciences offered a prize for a general proof of FLT. In 1857, the Academy awarded 3000 Francs and a gold medal to Kummer for his attempt of FLT which gave birth to the Theory of Ideal numbers. Another prize was offered in 1883 by the Academy of Brussels.

In 1908, a German industrialist and amateur mathematician Paul Wolfskehl bequeathed 100,000 gold marks, to the Gottingen Academy of Sciences, to offer as a prize for a complete proof of FLT. Thousands of incorrect proofs of FLT were submitted to the Gottingen Wolfskehl Prize Committee. Howard Eves once wrote, "FLT has the peculiar distinction of being the mathematical problem for which the greatest number of incorrect proofs published".

Great mathematicians like Euler, Legendre, Lebesgue, Kronecker, Cauchy, Lame, Dirichlet, Sophie Germain, Abel, Srinivasa Ramanujan and a host of others tried their hands. In many cases though their efforts did not produce that marvelous proof, they enriched number theory with newer branches, concepts, techniques and approaches. Centuries of glorious failure on one side, Godel's statement on the other side, made mathematicians to come to the conclusion that they might be searching for a non-existent solution. Yet some mathematicians continued their search.

Soon after the second world war, computers helped to prove the theorem for all values of n up to 500, then up to 1000, and then to 10,000. In 1980s Samuel S. Watstaff of the University of Illinois raised the limit to 25,000 and more recently mathematicians could claim that FLT was true for all values of N up to four million. In other words, for the first four million equations, mathematicians had proved that there were no numbers that filled any of them. "Even though the theorem had been proved for all values of N up to four million, there is no reason why it should be true for N= 4,000,001". And in the future the super computer proves the theorem for all values of N up to one Zillion , there is no reason why it should be true for N = One zillion and one.

Euler's equation $x^4 + y^4 + z^4 = w^4$ claimed that there are no whole number solutions to this equation. For 200 years nobody could prove Euler's Conjecture. But in 1988, Noam Elkies of Harvard University, discovered the following solution

$$(2,682,440)^4 + (15,365,639)^4 + (18,796,760)^4 = (20,615,673)^4$$

The Euler's conjecture turned out to be false. In fact Elkies proved that there are infinitely many solutions to this equation.

It is pertinent to note that David Hilbert announced 23 famous problems in the International Congress of Mathematicians held at Paris in the year 1900 and the list did not include FLT.

III

The Book by E.T. Bell- titled, *The Last Problem*, was published a year after the demise of its author. It was republished after 30 years thence, by the Mathematical

Association of America with an introduction by Underwood Dudley. Dudley states, "It is not a book of mathematics -pages go without an equation appearing and in mathematics books you are not told such things as the ancient Spartans were as virile as gorillas and as hard (including their heads) as bricks". History books do not contain nine equations in ten unknowns that come from the cattle problem of Archimedes...It is not a history of Number Theory.

It is a surprising coincidence: we recall that when Bell retired from the California Institute of Technology, Pasadena, CA, US , in 1953, he was presented with a copy of the Book of Diophantus, that is *Arithmetica* (1670) signed by many men and women.

V. Frederick Rickey, in one of his presentations, cited examples of John Nash, Andrew Wiles, Julia Robinson and others who were inspired by reading Bell. That is why he echoed the sentiment of Laplace, by declaring

Read Bell, Read Bell, He is the inspiration of us all

We recall that E.T. Bell believed that FLT would still be unresolved when human civilization gets destroyed by nuclear war. Bell made this prediction shortly before his own death in 1960. If he had lived longer, it would have been a delight to learn that his own book -*The Last Problem* inspired an adventurous and brilliant boy to attack FLT and solve.

This adventurous and brilliant boy was Andrew John Wiles. He was born in 1953 in Cambridge, England, the son of Maurice Frank Wiles, the Regius Professor of Divinity, at the University of Oxford, UK, and Patricia Wiles. Andrew went to King's College School, Cambridge during 1952-55. He went to Leys School at Cambridge also for some time. One day while coming home from his school, Young Wiles chanced to drop into the Library in Milton Road. Some puzzle books attracted his attention. Usually Books on Riddles, Conundrums and Puzzles would be packed with assortment of questions and solutions also would be provided for. But the Book that Wiles pulled out was with only one problem and no solution!

IV

Lagrange had no contribution to FLT, nor did he express any interest in the problem.

In 1876, Henry J.S. Smith, the then Sullivan Professor of Geometry at Oxford, delivering the Presidential (Outgoing) address to the London Mathematical Society, - (Topic was -On the Present Status of and Prospects of Some Branches of Pure mathematics), did not even mention about FLT.

FLT did not find a place in the syllabus of Felix Klein's (1849-1925), Seminar, in Gottingen.

At another point of time, Hilbert's comment regarding FLT is quite interesting. "Fortunately there is no mathematician except me, who is in a position to solve this question. However, I do not intend to slaughter myself the goose that lays golden eggs".

Again at the 1912 International Congress of Mathematicians held at Cambridge, UK, famous Number - Theorist of Gottingen, Germany, Edmund Landau (1877-1938), in his invited plenary lecture, titled, "Solved and Unsolved Problems of Number Theory", not even mentioned FLT.

Another instance, when Landau was giving a lecture on the occasion of the inauguration of the Hebrew University at Jerusalem in 1925, he dwelt upon the same theme and gave a list of 23 problems, as Hilbert did in 1900 conference of ICM, which did not obviously include FLT.

Fermat indicated that he had a proof for every one of his observations. So as far as he was concerned they were theorems. However Mathematicians, thereafter established the validity of the observations - whether they are true or false. Thus all results announced by Fermat were tackled one by one, but the statement about the equation $x^n + y^n = z^n$, for $n > 2$, stood stubbornly to give in. Therefore it acquired the appellation - Fermat Last Theorem, because it remained the last one of the results of Fermat to be proved.

While browsing rapidly through the pages of the Book by E. T. Bell, Wiles soon discovered that "apparently innocent looking equation, $x^2 + y^2 = z^2$ has a darker side. The Book described the existence of a mathematical monster" - The FLT.

The story of FLT revolved around the search for a missing proof. The Book that Wiles read for a while stated that the proof had been lost long ago. "There was no hint of what it might have been, no clues as to the Proof's construction or derivation. Wiles found himself puzzled, infuriated and intrigued".

Young Wiles was encouraged by his teacher in the school, who had done research in mathematics and he gave him a book about number theory that provided him some clues to start his attack on the problem. Wiles decided that he ought to study the vast literature about the unsuccessful attempts of earlier giants of the subject.

V

FLT displays a discrete charm of Mathematics.

The Book *The Last Problem* contained the biography of the famous problem. Wiles was fascinated by reading through a few pages of the Book. He was greatly surprised of the problem that was so easy to state, clearly understandable by a school boy of his age, but it defied solution for centuries. He thought over and decided why not he be the first one to prove it. From then on it was his "childhood dream and later obsession of his youth". However Wiles realized that his knowledge was not that sufficient to attack the problem; so he kept aside his childhood dream until it was brought back to his attention at the age of 33, while he was a Professional Mathematician at Princeton, New Jersey and came across the writings of professors like Gerhard Frey, Ken Ribet and such others.

Wiles got his Bachelor's degree in 1974 from Merton College, Oxford and Ph.D. in 1980 from Clare College, Cambridge under the thoughtful supervision of John Coates, who put Wiles on the right track by assigning him focused study on elliptic equations. These equations were originally studied by the ancient Greek mathematicians including Diophantus who wrote many details about these equations in his *Arithmetica*. These equations inspired Fermat and he welcomed the challenges they offered. Even after two thousand years these equations offered formidable problems. "By encouraging Wiles to study elliptic equations, Coates had given Wiles tools that would later enable him to work on his dream".

After acquainting himself with the writings of Gerhard Frey and Ken Ribet in 1986 Andrew Wiles realized that it might be possible to prove FLT, via Taniyama-Shimura conjecture. Over two decades elapsed since Andrew Wiles had discovered the Bell's Book for the first time in a library during his casual visit, but for the first time he could see vaguely a track towards realizing his childhood dream. Thanks to his research supervisor, Prof Coates, Wiles probably knew more about elliptic equations than anybody else. With his arsenal and firm determination, Wiles resumed his mission at Princeton University, US, where he became a professor. Most of the other mathematicians believed that hunting for the solution of FLT was a futile exercise. Even Wiles's Guru, Prof. Coates, said once- "I myself was skeptical that the beautiful link between FLT and the Taniyama-Shimura conjecture would actually lead to anything, because I must confess I did not think that the Taniyama-Shimura conjecture was accessible to proof. Beautiful though this problem was, it seemed impossible to actually prove. I should confess, I thought I probably would not see it proved in my life time".

VI

Against odds Andrew Wiles, started his battle royal, in all seriousness. He read almost all the recent journals and mastered the latest techniques, until they became a second nature to him. After 1986, for a period of 18 months, he mastered every bit of mathematics connected to elliptic equations or modular forms. He was prepared for a decade of single-minded *tapas*.

Wiles abandoned any work that was not directly relevant to proving FLT. He stopped attending lectures, conferences, and colloquia except those which formed part of his official responsibilities and obligations. Wiles made a decision to work in total isolation. He organized his study center on the attic of his home in Princeton University Campus. This is contrary to the modern culture of research work in mathematics where cooperation and collaboration are the driving forces of innovative findings. Wiles methodology was the practice of previous eras. It was as if he was imitating the style of his hero- Fermat himself.

In order to prove FLT, Wiles had to prove the Taniyama-Shimura conjecture *Every single elliptic equation can be correlated with a modular form.* Even before its link to the last theorem was guessed, mathematicians had tried desperately to prove the conjecture, but every attempt ended in failure. But Wiles did not give up. He said, "I carried this thought around in my head basically the whole time. I would wake up with it first thing in the morning. I would be thinking about it all the day, and I would be thinking about it when I went to sleep, without distraction. I would have the same thing going round and round in my mind".

On March 8, 1988, Wiles read from headline news in the *New York Times and The Washington Post,* claiming that one, Dr. Yoichi Miyaoka of Tokyo Metropolitan University, had found a solution to FLT and was shocked. Miyaoka's method was solving FLT, through differential geometric approach. As was Wiles's goal of solving FLT through first proving the conjecture of Taniyama-Shimura, so was Miyaoka's to solve FLT, via proving a conjecture known as Miyaoka's Inequality. As the final result was to be reached through his own conjecture, everybody thought that Miyaoka would win the battle. Miyaoka announced his final result and scrutiny began on his work. It was found that there was a worrying contradiction in the proof and the 350 year old puzzle remained unsolved. Of course, Wiles breathed a sigh of relief and "FLT remained unconquered and he could continue his battle to prove it via Taniyama-Shimura conjecture".

After three years of non-stop combat, Wiles made a series of breakthroughs and pursued his efforts for another year. "He began working on another technique called *Iwasawa Theory.* It is a method of analysing elliptic equations that he had learnt as research scholar with John Coates".

After some progress, making use of Galois groups, Wiles was again frustrated. In the summer of 1991, he came to the conclusion that he almost failed in his endeavour to adapt *Iwasawa Theory.* He searched exhaustively in the literature and was unable to find an alternative technique. At that point of time, he strongly felt that it would benefit him, if he attended the then forthcoming Boston Conference on elliptic equations. John Coates was also attending the Conference. Coates mentioned to Wiles that there was a student of his by name- Matheus Flach writing a paper on elliptic equations using a recent method of Kolyvagin. Wiles guessed that that method might help him. Then he devoted day and night for several months in extending Kolyvagin-Flach method to suit to his requirement. Week by week he was making progress. Then in Jan 1993, he decided to confide with Prof Nick Katz, a close friend of his, to help him in checking the extensive calculations of the massive work turned out.

By May 1993, Wiles was convinced that the whole of FLT was in his hands. At the end of June, a conference was coming up in Cambridge UK, the venue being Isaac Newton's Institute. John Coates was one of the organisers of the event.

Wiles reached Cambridge two and half weeks before the dates of his lectures. He got Kolyvagin-Flach part checked thoroughly by Prof Barry Mazur of Harvard University. Wiles asked for three slots for his presentation and accordingly got. The title of Wiles Lecture series was *Modular Forms, Elliptic Curves and Galois Representations*

We can note that the title of the lectures was vague that it gave no hint of his ultimate aim. Wiles in his first lecture Monday, June 21, 1993, laid foundation for his proving the Taniyama-Shimura Conjecture in the next day talk. In the second day talk, he did some calculations related to the conjecture that was coming up. People in the hall started guessing and whispering where Wiles was leading them.

On June 23, Andrew gave the final talk. At the concluding part, he just wrote up the statement of FLT and said, "I think I will stop here". There was sustained applause for a long time. It was a lecture of the century!

Within hours of Wiles original statement, Karl Rubin of Ohio State University and Ken Ribet of Berkeley, who were present at the Cambridge Conference, sent separate e-mails to people throughout the mathematical community, giving the brief sketches of the proof.

Wiles when asked what it feels like to solve a puzzle, that has mystified mathematicians for centuries, he said, "It is thrilling. It is the experience we live for, this insight, that suddenly you see that everything clearly before you that has been so abstruse and so frustrating for so long".

Subsequently some problems arose and again Wiles went through trying times. Fortunately for Wiles they were resolved in good time with the help of his own student Richard Taylor.

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